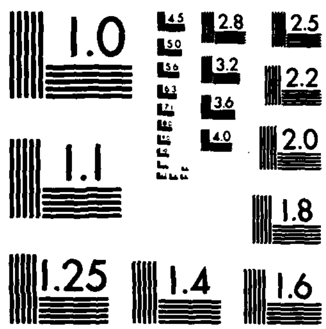
AD-A158 740 BASIC PROPERTIES OF STRONG MIXING CONDITIONS(U) NORTH CAROLINA UNIV AT CHAPEL HILL CENTER FOR STOCHASTIC PROCESSES R C BRADLEY JUN 85 TR-102 AFOSR-TR-85-0619 1/1

UNCLASSIFIED F49620-82-C-0009 F/G 12/1 NL

END

FILMED

DTIC



MICROCOPY RESOLUTION TEST CHART

DARDS-1963-A

CENTER FOR STOCHASTIC PROCESSES

Department of Statistics
University of North Carolina
Chapel Hill, North Carolina

BASIC PROPERTIES OF STRONG MIXING CONDITIONS

by

Richard C. Bradley

TECHNICAL REPORT 102

June 1985

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION: UNCLASSIFIED

1b. RESTRICTIVE MARKINGS: AD-A158 740

2a. SECURITY CLASSIFICATION AUTHORITY:

2b. DECLASSIFICATION/DOWNGRADING SCHEDULE:

3. DISTRIBUTION/AVAILABILITY OF REPORT: Approved for public release; distribution unlimited.

4. PERFORMING ORGANIZATION REPORT NUMBER(S): Technical Report 102

5. MONITORING ORGANIZATION REPORT NUMBER(S): AFOSR-TR- 85-0619

6a. NAME OF PERFORMING ORGANIZATION: Center for Stochastic Processes

6b. OFFICE SYMBOL (If applicable):

7a. NAME OF MONITORING ORGANIZATION: Air Force Office of Scientific Research

6c. ADDRESS (City, State and ZIP Code): University of North Carolina, Statistics Department, 321 PH 039A, UNC, Chapel Hill, NC 27413

7b. ADDRESS (City, State and ZIP Code): Bolling AFB, Washington, DC 20332

8a. NAME OF FUNDING/SPONSORING ORGANIZATION: AFOSR

8b. OFFICE SYMBOL (If applicable): NM

9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER: F 49620 82 C 0009

8c. ADDRESS (City, State and ZIP Code): Bolling AFB, Washington, DC 20332

10. SOURCE OF FUNDING NOS.

PROGRAM ELEMENT NO.	PROJECT NO.	TASK NO.	WORK UNIT NO.
61102F	2304	A5	

11. TITLE (Include Security Classification): BASIC PROPERTIES OF STRONG MIXING CONDITIONS

12. PERSONAL AUTHOR(S): Richard C. Bradley

13a. TYPE OF REPORT: technical

13b. TIME COVERED: FROM 9/84 TO 8/85

14. DATE OF REPORT (Yr., Mo., Day): June 1985

15. PAGE COUNT: 37

16. SUPPLEMENTARY NOTATION:

17. COSATI CODES

FIELD	GROUP	SUB. GR.

18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number): Key words: strictly stationary, strong mixing conditions, Markov chains, Gaussian processes, measures of dependence.

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

This is a survey of the basic properties of strong mixing conditions for sequences of random variables. The focus will be on the "structural" properties of these conditions, and not at all on limit theory. For a discussion of central limit theorems and related results under these conditions, the reader is referred to Peligrad [60] or Iosifescu [50]. This survey will be divided into eight sections, as follows:

1. Measures of dependence, 2. Five strong mixing conditions, 3. Mixing conditions for two or more sequences, 4. Mixing conditions for Markov chains, 5. Mixing conditions for Gaussian sequences, 6. Some other special examples, 7. The behavior of the dependence coefficients, 8. Approximation of mixing sequences by other random sequences.

20. DISTRIBUTION/AVAILABILITY OF ABSTRACT: UNCLASSIFIED/UNLIMITED ☒ SAME AS RPT. ☐ DTIC USERS ☐

21. ABSTRACT SECURITY CLASSIFICATION: UNCLASSIFIED

22a. NAME OF RESPONSIBLE INDIVIDUAL: Brian W. Woodruff, Maj, USAF; Jane Wille

22b. TELEPHONE NUMBER (Include Area Code): 767-5027; 919 962-2307

22c. OFFICE SYMBOL: NM

DD FORM 1473, 83 APR — EDITION OF 1 JAN 73 IS OBSOLETE.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE

BASIC PROPERTIES OF STRONG MIXING CONDITIONS

by

Richard C. Bradley
Department of Mathematics
Indiana University
Bloomington
Indiana 47405

and

Center for Stochastic Processes
Department of Statistics
University of North Carolina
Chapel Hill, North Carolina 27514

This work was partially supported by AFOSR Grant No. F49620 82 C 0009 and by NSF Grant No. DMS 84-01021.

This is a survey of the basic properties of strong mixing conditions for sequences of random variables. The focus will be on the "structural" properties of these conditions, and not at all on limit theory. For a discussion of central limit theorems and related results under these conditions, the reader is referred to Peligrad [60] or Iosifescu [50]. This survey will be divided into eight sections, as follows:

1) Measures of dependence,
2) Five strong mixing conditions,
3) Mixing conditions for two or more sequences,
4) Mixing conditions for Markov chains,
5) Mixing conditions for Gaussian sequences,
6) Some other special examples,
7) The behavior of the dependence coefficients, and
8) Approximation of mixing sequences by other random sequences

1. MEASURES OF DEPENDENCE

Suppose $(\Omega,\mathcal{F},P)$ is a probability space. For any two σ-fields $\mathcal{A}$ and $\mathcal{B} \subset \mathcal{F}$ define the following measures of dependence:

$$\alpha(\mathcal{A},\mathcal{B}) := \sup |P(A \cap B) - P(A)P(B)|, \ A \in \mathcal{A}, \ B \in \mathcal{B}. \tag{1.1}$$

$$\phi(\mathcal{A},\mathcal{B}) := \sup |P(B|A) - P(B)|, \ A \in \mathcal{A}, \ B \in \mathcal{B}, \ P(A) > 0. \tag{1.2}$$

$$\phi_{rev}(\mathcal{A},\mathcal{B}) := \phi(\mathcal{B},\mathcal{A}) \quad (\text{"rev" stands for "reversed"}). \tag{1.3}$$

$$\psi(\mathcal{A},\mathcal{B}) := \sup \frac{|P(A\cap B) - P(A)P(B)|}{P(A)P(B)}, \ A \in \mathcal{A}, \ B \in \mathcal{B}. \tag{1.4}$$

$$\rho(\mathcal{A},\mathcal{B}) := \sup |\mathrm{Corr}(X,Y)|, \ X \in L_2(\mathcal{A}), \ Y \in L_2(\mathcal{B}); \ X,Y \text{ real}. \tag{1.5}$$

$$\beta(\mathcal{A},\mathcal{B}) := \sup \tfrac{1}{2}\Sigma_{i=1}^{I}\Sigma_{j=1}^{J}|P(A_i \cap B_j) - P(A_i)P(B_j)|. \tag{1.6}$$

where this latter sup is taken over all pairs of partitions $\{A_1,\ldots, A_I\}$ and $\{B_1,\ldots, B_J\}$ of Ω such that $A_i \in \mathcal{A}$ for all i and $B_j \in \mathcal{B}$ for all j. In (1.4) and in the sequel, 0/0 is interpreted to be 0. These measures of dependence will be the basis for the mixing conditions that we shall study, starting with Section 2. Here in Section 1 we shall just study these measures of dependence.

The following inequalities hold:

$$2\alpha(\mathcal{A},\mathcal{B}) \le \beta(\mathcal{A},\mathcal{B}) \le \phi(\mathcal{A},\mathcal{B}) \le \tfrac{1}{2}\psi(\mathcal{A},\mathcal{B}). \tag{1.7}$$

$$4\alpha(\mathcal{A},\mathcal{B}) \le \rho(\mathcal{A},\mathcal{B}) \le \psi(\mathcal{A},\mathcal{B}). \tag{1.8}$$

$$\rho(\mathcal{A},\mathcal{B}) \le 2\phi^{\frac{1}{2}}(\mathcal{A},\mathcal{B})\cdot\phi_{rev}^{\frac{1}{2}}(\mathcal{A},\mathcal{B}). \tag{1.9}$$

$$\rho(\mathcal{A},\mathcal{B}) = \sup \|E(F|\mathcal{B}) - Ef\|_2/\|f\|_2, \ f \in L_2(\mathcal{A}), \ f \text{ real}. \tag{1.10}$$

$$\alpha(\mathcal{A},\mathcal{B}) \le \tfrac{1}{4}, \ \beta(\mathcal{A},\mathcal{B}) \le 1, \ \phi(\mathcal{A},\mathcal{B}) \le 1, \ \rho(\mathcal{A},\mathcal{B}) \le 1. \tag{1.11}$$

Eqn. (1.9), an improvement of the earlier well known inequality $\rho(\mathcal{A},\mathcal{B}) \le 2\phi^{\frac{1}{2}}(\mathcal{A},\mathcal{B})$, comes from Peligrad [59, p. 462, eqn. (4)]; independently the kindred inequality $\rho(\mathcal{A},\mathcal{B}) \le 2\cdot\max\{\phi(\mathcal{A},\mathcal{B}), \phi_{rev}(\mathcal{A},\mathcal{B})\}$ was given by Denker and Keller [34, p. 516, line -8]. In this last inequality as well as in

(1.7), (1.8), and (1.9), equality is achieved in some simple cases such as when $\mathcal{A} = \mathcal{B} = \{\Omega, A, A^c, \phi\}$ where $P(A) = \frac{1}{2}$. Eqns. (1.7), (1.8), (1.10), and (1.11) are all either trivial or at least fairly easy to prove. Referring to eqn. (1.11), $\psi(\mathcal{A},\mathcal{B})$ can of course take on the value $+\infty$. Each of the measures of dependence in eqns. (1.1)-(1.6) takes the value 0 precisely when $\mathcal{A}$ and $\mathcal{B}$ are independent σ-fields.

The measures of dependence in eqns. (1.1)-(1.6) fit nicely into a more general framework using "norms" of the bilinear form "covariance". For any σ-field $\mathcal{A}$, let $\mathcal{S}(\mathcal{A})$ denote the set of all complex-valued simple $\mathcal{A}$-measurable random variables. (The particularly nice form of eqn. (1.13) below depends on the use of complex-valued rather than just real-valued random variables; however, this is not of any special importance.) Define the following families of measures of dependence between pairs of σ-fields $\mathcal{A}$ and $\mathcal{B}$:

For $0 \le r,s \le 1$,

$$\alpha_{r,s}(\mathcal{A},\mathcal{B}) := \sup \frac{|P(A\cap B) - P(A)P(B)|}{[P(A)]^r [P(B)]^s}, \quad A \in \mathcal{A},\ B \in \mathcal{B}.$$

For $1 \le p,q \le \infty$,

$$R_{p,q}(\mathcal{A},\mathcal{B}) := \sup \frac{|EXY - EXEY|}{\|X\|_p \|Y\|_q}, \quad X \in \mathcal{S}(\mathcal{A}),\ Y \in \mathcal{S}(\mathcal{B}).$$

(Note that $\alpha_{r,s}(\cdot,\cdot)$ is a variant of $R_{1/r,1/s}(\cdot,\cdot)$ using only indicator functions.) Obviously the measures of dependence in eqns. (1.1)-(1.5) are respectively $\alpha_{0,0}(\mathcal{A},\mathcal{B})$, $\alpha_{1,0}(\mathcal{A},\mathcal{B})$, $\alpha_{0,1}(\mathcal{A},\mathcal{B})$, $\alpha_{1,1}(\mathcal{A},\mathcal{B})$, and $R_{2,2}(\mathcal{A},\mathcal{B})$. (The equation $\rho(\mathcal{A},\mathcal{B}) = R_{2,2}(\mathcal{A},\mathcal{B})$ holds by [75, Theorem 1.1] and a simple calculation.) Also it is easy to show that if one modifies the definition

of $R_{\infty,\infty}(\mathcal{A},\mathcal{B})$ in an appropriate way so as to allow the r.v.'s X and Y to take their values in the Banach spaces ℓ^1 and ℓ^∞ respectively, then one obtains a measure of dependence that is within a positive constant factor of $\beta(\mathcal{A},\mathcal{B})$ in eqn. (1.6) (see [22, Section 2.2]).

If $0 \le r_0, r_1, s_0, s_1 \le 1$, $0 < \theta < 1$, $r := (1-\theta)r_0 + \theta r_1$, and $s := (1-\theta)s_0 + \theta s_1$, then for any two σ-fields $\mathcal{A}$ and $\mathcal{B}$,

$$\alpha_{r,s}(\mathcal{A},\mathcal{B}) \le [\alpha_{r_0,s_0}(\mathcal{A},\mathcal{B})]^{1-\theta} \cdot [\alpha_{r_1,s_1}(\mathcal{A},\mathcal{B})]^{\theta}, \tag{1.12}$$

$$R_{1/r,1/s}(\mathcal{A},\mathcal{B}) \le [R_{1/r_0,1/s_0}(\mathcal{A},\mathcal{B})]^{1-\theta} \cdot [R_{1/r_1,1/s_1}(\mathcal{A},\mathcal{B})]^{\theta}. \tag{1.13}$$

Eqn. (1.12) has a trivial one-line proof and is useful for comparing the various measures of dependence $\alpha_{r,s}$ (see [21, Theorems 3.1, 3.2, and 4.1 (i)(ii)]). Eqn. (1.13) is an application of Thorin's multilinear version of the Riesz-Thorin interpolation theorem (see e.g. [5, p. 18, Exercise 13]); eqn. (1.13) and variants of it are useful for comparing various measures of dependence and for studying the relations between them (see [67, Chapter 7] [56, Lemma 1] [21] [22]). For example, as a consequence of (1.13) one can show that if $1 \le p,q,t \le \infty$ and $1/p + 1/q + 1/t = 1$, then for any two σ-fields $\mathcal{A}$ and $\mathcal{B}$,

$$R_{p,q}(\mathcal{A},\mathcal{B}) \le (2\pi) \cdot [\alpha(\mathcal{A},\mathcal{B})]^{1/t} \cdot [\phi(\mathcal{A},\mathcal{B})]^{1/p} \cdot [\phi_{rev}(\mathcal{A},\mathcal{B})]^{1/q}$$

(see [21, Theorem 1.1(i)]). Except for a constant factor, this inequality covers some other previously known ones as special cases, including eqn. (1.9). In an obvious way, a "small" upper bound on $R_{p,q}(\mathcal{A},\mathcal{B})$ might lead to a "small" value of Cov(X,Y) if, say, X and Y are r.v.'s which are $\mathcal{A}$-measurable and $\mathcal{B}$-measurable respectively. Such bounds are often useful in the proofs of limit theorems for dependent random variables.

Further information about measures of dependence can be gained from the use of other methods and results in interpolation theory, such as the techniques in the Marcinkiewicz interpolation theorem and the Stein-Weiss [72] methods for handling indicator functions. That observation is due to W. Bryc. For example, using such techniques Bryc proved that if $1 < p,q < \infty$ and $1/p + 1/q = 1$, then for any two σ-fields $\mathcal{A}$ and $\mathcal{B}$,

$$R_{p,q}(\mathcal{A},\mathcal{B}) \le C \cdot \alpha_{1/p,1/q}(\mathcal{A},\mathcal{B}) \cdot [1 - \log \alpha_{1/p,1/q}(\mathcal{A},\mathcal{B})] \tag{1.14}$$

where C is a positive constant that depends only on p and q (see [21, Theorem 4.1(vi)]). (For the case $p = q = 2$, (1.14) improved a similar but much weaker inequality established in [15] by different methods; see also [24].) For any choice of p and q meeting the given specifications, (1.14) is within a constant factor of being sharp (see [22, Section 1.1]).

Let us say that two measures of dependence are "equivalent" if each one becomes arbitrarily small as the other one becomes sufficiently small. Among the measures of dependence $\alpha_{r,s}$, $0 \le r,s \le 1$, and $R_{p,q}$, $1 \le p,q \le \infty$, there are only five equivalence classes which consist of more than one member:

(i) $\alpha_{r,s}$, $r + s < 1$, $R_{p,q}$, $1/p + 1/q < 1$;

(ii) $\alpha_{r,1-r}$, $0 < r < 1$, $R_{p,p'}$, $1 < p < \infty$ (where p' is defined by $1/p + 1/p' = 1$);

(iii) $\alpha_{1,0}$, $R_{1,\infty}$;

(iv) $\alpha_{0,1}$, $R_{\infty,1}$;

(v) $\alpha_{1,1}$, $R_{1,1}$ (which are equal by a simple argument).

These five classes each contain one of the measures of dependence in eqns. (1.1)-(1.5). The measure β in (1.6) is not equivalent to any of the measures $\alpha_{r,s}$ or $R_{p,q}$. All of this is explained in [21, Remark 4.1] and [22, Sections 1.2 and 2.2].

2. FIVE STRONG MIXING CONDITIONS

Henceforth all random variables are real-valued. For any family $(Y_s, s \in S)$ of random variables (where S is an index set), the notation $\sigma(Y_s, s \in S)$ will mean the σ-field generated by this family, i.e. the smallest σ-field containing the events $\{Y_s < r\}$, $s \in S$, $r \in \mathbb{R}$.

Suppose $(X_k, k \in \mathbb{Z})$ is a sequence of random variables. (No assumption of stationarity is made yet.) For $-\infty \le J \le L \le \infty$ define $F_J^L := \sigma(X_k, J \le k \le L)$. Referring to eqns. (1.1)-(1.6), for each $n = 1,2,3,\ldots$ define

$\alpha(n) := \sup_{J \in \mathbb{Z}} \alpha(F_{-\infty}^J, F_{J+n}^\infty)$,

$\phi(n) := \sup_{J \in \mathbb{Z}} \phi(F_{-\infty}^J, F_{J+n}^\infty)$,

$\psi(n) := \sup_{J \in \mathbb{Z}} \psi(F_{-\infty}^J, F_{J+n}^\infty)$,

$\rho(n) := \sup_{J \in \mathbb{Z}} \rho(F_{-\infty}^J, F_{J+n}^\infty)$,

$\beta(n) := \sup_{J \in \mathbb{Z}} \beta(F_{-\infty}^J, F_{J+n}^\infty)$.

The sequence (X_k) is said to be

strongly mixing [66] if $\lim_{n\to\infty} \alpha(n) = 0$,

ϕ-mixing [46] if $\lim_{n\to\infty} \phi(n) = 0$,

ψ-mixing ([8] essentially) if $\lim_{n\to\infty} \psi(n) = 0$,

ρ-mixing [52] if $\lim_{n\to\infty} \rho(n) = 0$,

absolutely regular [73] if $\lim_{n\to\infty} \beta(n) = 0$.

These are the five conditions on which we shall focus. The ψ-mixing condition actually evolved from the "*-mixing" condition, which was the condition studied in [8]: $\lim_{n\to\infty} \sup_J \psi(F_{-\infty}^J, F_{J+n}^{J+n}) = 0$. The maximal correlation coefficient $\rho(A,B)$ was studied in [44] [39], much earlier than the ρ-mixing condition. The absolute regularity condition was attributed in [73] to Kolmogorov.

Several minor comments are in order:

(i) In defining the strong mixing condition ($\alpha(n) \to 0$) for a "singly-infinite" sequence (X_k, $k = 1,2,3,\ldots$) one modifies the definition of $\alpha(n)$ as follows: $\alpha(n) := \sup_{J\geq 1}\alpha(\mathcal{F}_1^J, \mathcal{F}_{J+n}^{\infty})$.

(ii) In defining the strong mixing condition for a *strictly stationary* doubly-infinite sequence (X_k, $k \in \mathbb{Z}$) one can simply define $\alpha(n)$ by $\alpha(n) := \alpha(\mathcal{F}_{-\infty}^0, \mathcal{F}_n^{\infty})$.

(iii) In whatever context one is dealing with, the sequence of numbers $\alpha(1), \alpha(2), \alpha(3), \ldots$ is obviously automatically non-increasing.

(iv) If a given random sequence (X_k, $k \in \mathbb{Z}$) is strongly mixing, and for each $k \in \mathbb{Z}$, $f_k: \mathbb{R} \to \mathbb{R}$ is a Borel-measurable function, then the random sequence ($f_k(X_k)$, $k \in \mathbb{Z}$) is also obviously strongly mixing, with the dependence coefficients $\alpha(n)$, $n = 1,2,\ldots$ for the sequence $(f_k(X_k))$ being no greater than the corresponding ones for (X_k).

(v) If a *strictly stationary* strongly mixing singly-infinite sequence $(X_1, X_2, X_3, \ldots)$ is extended to a *strictly stationary* doubly-infinite sequence (X_k, $k \in \mathbb{Z}$), then this new doubly-infinite sequence is also strongly mixing, with precisely the same dependence coefficients $\alpha(n)$, $n = 1,2,\ldots$.

(vi) Comments (i)-(v) carry over verbatim to the other mixing conditions defined above (ϕ-mixing, ψ-mixing, ρ-mixing, and absolute regularity) and their dependence coefficients.

By eqns. (1.7), (1.8), (1.9), and (1.11) the following implications hold for a given random sequence:

(i) ρ-mixing $\Rightarrow$ strong mixing.

(ii) absolute regularity $\Rightarrow$ strong mixing.

(iii) ϕ-mixing $\Rightarrow$ ρ-mixing and absolute regularity.

(iv) ψ-mixing $\Rightarrow$ ϕ-mixing.

Among these five mixing conditions there are (aside from transitivity) no other general implications. (For special families of random sequences, however, e.g. Gaussian sequences, discrete Markov chains, etc., there are other implications; this will be seen in more detail in Sections 4 and 5 later on.) Since "strong mixing" is the weakest of these five conditions, these conditions -- and others that imply strong mixing -- are sometimes referred to collectively as "strong mixing conditions" (plural). The term "strong mixing condition" (singular) will refer to the condition $\alpha(n) \to 0$ as above. Of course all of these mixing conditions are satisfied by sequences of independent r.v.'s and also by m-dependent sequences. Other examples will be encountered in Sections 4, 5, 6, and 7 later on. Later in Section 2 here the strong mixing condition will be compared to standard conditions in ergodic theory.

For a given sequence $(X_k, k \in \mathbb{Z})$ the ϕ-mixing condition is not necessarily preserved if the direction of "time" is reversed. Referring to eqn. (1.3), define for each $n = 1,2,3,\ldots$, $\phi_{rev}(n) = \sup_{J\in\mathbb{Z}} \phi_{rev}(F_{-\infty}^{J}, F_{J+n}^{\infty}) = \sup_{J\in\mathbb{Z}} \phi(F_{J+n}^{\infty}, F_{-\infty}^{J})$. In [51, p. 414] there is an example of a strictly stationary countable-state Markov chain $(X_k, k \in \mathbb{Z})$ such that $\phi(n) \to 0$ as $n \to \infty$ and $\phi_{rev}(n) = 1$ for all $n \geq 1$. "Symmetric" versions of the ϕ-mixing condition, putting equal emphasis on $\phi(n)$ and $\phi_{rev}(n)$, have been useful in limit theory (see [34] [59]).

In the rest of Section 2, and also in most of the rest of this paper, we shall deal only with a strictly stationary doubly-infinite sequence $(X_k, k \in \mathbb{Z})$.

For measure-theoretic convenience, for the rest of Section 2 we shall assume that our probability space is $(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\mathbb{Z}}, P)$. (In a context such as

this, the symbol $\mathcal{B}$ is intended to mean the standard Borel σ-field.) We shall of course assume that for each $\omega \in \mathbb{R}^{\mathbb{Z}}$ and each $k \in \mathbb{Z}$, $X_k(\omega) := \omega_k$. (The notation ω_k means the k^{th} coordinate of ω.) We shall use a regular conditional distribution of $(X_1, X_2, X_3, \ldots)$ given $(X_0, X_{-1}, X_{-2}, \ldots)$. In this context a standard measure-theoretic argument will show that, under our assumption that (X_k) is strictly stationary, for each $n \geq 1$,

$$\phi(n) = \text{ess sup}[\sup|P(B|\mathcal{F}_{-\infty}^{0}) - P(B)|,\ B \in \mathcal{F}_{n}^{\infty}],$$

and

$$\beta(n) = E[\sup|P(B|\mathcal{F}_{-\infty}^{0}) - P(B)|,\ B \in \mathcal{F}_{n}^{\infty}].$$

There is another useful formulation of $\beta(n)$. Let Q denote the probability measure on $(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\mathbb{Z}})$ such that (i) under Q the σ-fields $\mathcal{F}_{-\infty}^{0}$ and $\mathcal{F}_{1}^{\infty}$ are independent and (ii) on each of these two σ-fields $\mathcal{F}_{-\infty}^{0}$ and $\mathcal{F}_{1}^{\infty}$ the measure Q is identical to P. Then (under our assumption of strict stationarity),

$$\begin{aligned} \beta(n) &= \sup|P(D) - Q(D)|,\ D \in \mathcal{F}_{-\infty}^{0} \vee \mathcal{F}_{n}^{\infty} \\ &= \tfrac{1}{2}\|P_n - Q_n\| \end{aligned} \tag{2.1}$$

where P_n (resp. Q_n) is the restriction of P (resp. Q) to $\mathcal{F}_{-\infty}^{0} \vee \mathcal{F}_{n}^{\infty}$ and $\|\cdot\|$ denotes total variation.

Let T denote the usual shift operator on $\mathbb{R}^{\mathbb{Z}}$; that is, for each $\omega \in \mathbb{R}^{\mathbb{Z}}$, $T\omega$ is defined by $(T\omega)_k := \omega_{k+1}$ for all $k \in \mathbb{Z}$. For any event $A \in \mathcal{F}_{-\infty}^{\infty}$ $(= \mathcal{B}^{\mathbb{Z}})$ we use the notation $TA := \{\omega: T^{-1}\omega \in A\}$. Our (strictly stationary) sequence (X_k) is said to be "mixing", or "mixing in the ergodic-theoretic sense", if for all $A, B \in \mathcal{F}_{-\infty}^{\infty}$, $\lim_{n\to\infty} P(A \cap T^n B) = P(A)\cdot P(B)$. (In ergodic theory this condition is sometimes referred to as "strong mixing", but we shall use the term "strong mixing" for the condition $\alpha(n) \to 0$ as before.) Our sequence (X_k) is said to be "regular" if its past tail σ-field $\bigcap_{n=1}^{\infty} \mathcal{F}_{-\infty}^{-n}$ is trivial

(i.e. contains only events of probability 0 or 1). It is well known that

(i) mixing (in the ergodic-theoretic sense) $\Rightarrow$ ergodic,

(ii) regular $\Rightarrow$ mixing (in the ergodic-theoretic sense), and

(iii) strong mixing $(\alpha(n) \to 0)$ $\Rightarrow$ regular.

Statements (ii) and (iii) are easy consequences of [47, p. 302, Theorem 17.1.1]. Naturally, in (ii) and (iii) one can replace "regular" by the condition that the future tail σ-field $\bigcap_{n=1}^{\infty} F_n^{\infty}$ be trivial. (In Example 6.2 in Section 6 we shall encounter a well known stationary regular sequence whose future tail σ-field fails to be trivial.)

If (X_k) is strictly stationary and absolutely regular, then its double tail σ-field $\bigcap_{n=1}^{\infty}(F_{-\infty}^{-n} \vee F_n^{\infty})$ is trivial (i.e. $P(D) = 0$ or 1 for every D in the double tail σ-field). This holds by (2.1) and an elementary measure-theoretic argument (one can use e.g. [74, Lemma 4.3]). In [19] a strictly stationary ρ-mixing sequence is constructed for which the double tail σ-field fails to be trivial.

Let us briefly give references for several other related mixing conditions, for strictly stationary sequences. The "information regularity" condition (see [65]) is like the strong mixing conditions defined above, using the "coefficient of information" as the basic measure of dependence. A "Cesaro" variant of strong mixing, known as "uniform ergodicity", was studied by Cogburn [25]; and Rosenblatt [68, Theorem 2] established a nice connection between this condition and the strong mixing condition itself. Another mixing condition weaker than strong mixing has played a nice role in extreme value theory (see e.g. [54]) as well as in convergence in distribution to non-normal stable laws (see [29]). A mixing condition based on characteristic functions was studied in [75]. Finally, by a theorem of Ornstein, a condition of weak dependence known as the "very

weak Bernoulli" condition characterizes the strictly stationary finite-state sequences that are isomorphic to a Bernoulli shift. For more information on the very weak Bernoulli condition, including recent generalizations of it to stationary real sequences in connection with central limit theory, see [71] [36] [32] [17] and the references therein.

3. MIXING CONDITIONS FOR TWO OR MORE SEQUENCES

Suppose $(X_k, k \in \mathbb{Z})$ and $(Y_k, k \in \mathbb{Z})$ are strongly mixing sequences that are independent of each other. Then the sequence of random vectors $((X_k, Y_k), k \in \mathbb{Z})$ is strongly mixing. Hence the sequence of sums $(X_k + Y_k, k \in \mathbb{Z})$ is also strongly mixing. The same comments apply to the other mixing conditions being discussed here. Pinsker [65, p. 73] noted this for absolute regularity. Under natural extra restrictions, such comments can be extended from two to countably many sequences that are independent of each other. Here we shall just present the basic propositions from which all of these comments can easily be deduced.

The first result is due to Csaki and Fischer [27, p. 40, Theorem 6.2]:

Theorem 3.1 (Csaki and Fischer): Suppose $\mathcal{A}_n$ and $\mathcal{B}_n$, $n = 1,2,3,\ldots$ are σ-fields and the σ-fields $(\mathcal{A}_n \vee \mathcal{B}_n)$, $n = 1,2,3,\ldots$ are independent. Then

$$\rho\Big(\bigvee_{n=1}^{\infty} \mathcal{A}_n, \bigvee_{n=1}^{\infty} \mathcal{B}_n\Big) = \sup_{n\geq 1} \rho(\mathcal{A}_n, \mathcal{B}_n).$$

For a short proof see Witsenhausen [76, Theorem 1]. In Example 4.4 in the next section an interesting application of Theorem 3.1 will be given. For the other dependence coefficients, slightly weaker statements hold:

Theorem 3.2: If the hypothesis of Theorem 3.1 is satisfied, then the following statements hold:

(i) $\alpha\Big(\bigvee_{n=1}^{\infty} \mathcal{A}_n, \bigvee_{n=1}^{\infty} \mathcal{B}_n\Big) \leq \sum_{n=1}^{\infty} \alpha(\mathcal{A}_n, \mathcal{B}_n)$.

(ii) $\beta\Big(\bigvee_{n=1}^{\infty} \mathcal{A}_n, \bigvee_{n=1}^{\infty} \mathcal{B}_n\Big) \leq \sum_{n=1}^{\infty} \beta(\mathcal{A}_n, \mathcal{B}_n)$.

(iii) $\phi\left(\bigvee_{n=1}^{\infty} A_n, \bigvee_{n=1}^{\infty} B_n\right) \le \sum_{n=1}^{\infty} \phi(A_n, B_n)$.

(iv) $\psi\left(\bigvee_{n=1}^{\infty} A_n, \bigvee_{n=1}^{\infty} B_n\right) \le \left(\prod_{n=1}^{\infty} [1 + \psi(A_n, B_n)]\right) - 1$.

Statements (i)-(iii) can be found in [12, Lemma 8], [20, Lemma 2.2], and [11, Lemma 2.2]. Statements (ii)-(iii) can also be derived easily from [37, Lemma 1]. Statement (iv) is an elementary consequence of [14, Lemma 1].

4. MIXING CONDITIONS FOR MARKOV CHAINS

Here a brief discussion is given of Markov chains satisfying strong mixing conditions. For a more thorough discussion of this topic see Rosenblatt [67, Chapter 7].

The following theorem is fundamental to the study of mixing conditions on Markov chains:

Theorem 4.1. Suppose $(X_k, k \in \mathbb{Z})$ is a strictly stationary real Markov chain. Then for all $n \geq 1$ the following five statements hold:

(i) $\alpha(n) = \alpha(\sigma(X_0), \sigma(X_n))$.

(ii) $\rho(n) = \rho(\sigma(X_0), \sigma(X_n))$.

(iii) $\beta(n) = \beta(\sigma(X_0), \sigma(X_n))$.

(iv) $\phi(n) = \phi(\sigma(X_0), \sigma(X_n))$.

(v) $\psi(n) = \psi(\sigma(X_0), \sigma(X_n))$.

The proof is an elementary measure-theoretic exercise using the Markov property. For example, see [8, Lemma 8] for a proof of (v). (Thus for Markov chains, ψ -mixing is equivalent to the "*-mixing" condition studied in [8]). As a consequence of (iv), for Markov chains the ϕ-mixing condition is equivalent to Doeblin's condition (see [67, p. 212, eqn. (18)]).

For the next theorem we shall use the following terminology: A sequence of non-negative numbers $a_1, a_2, a_3, \ldots$ is said to "converge to 0 exponentially fast" if there exists a positive number r such that $a_n = O(e^{-rn})$ as $n \to \infty$.

Theorem 4.2. Suppose $(X_k, k \in \mathbb{Z})$ is a strictly stationary real Markov chain. Then the following three statements hold:

(i) If $\rho(n) \to 0$, then $\rho(n) \to 0$ exponentially fast.

(ii) If $\phi(n) \to 0$, then $\phi(n) \to 0$ exponentially fast.

(iii) If $\psi(n) \to 0$, then $\psi(n) \to 0$ exponentially fast.

Part (iii) was proved in [8, pp. 8-9, Theorem 5]. The arguments for parts (i) and (ii) are similar. (For part (i) a simple argument using (1.10) and the Markov property will show the well known inequality $\rho(m+n) \le \rho(m) \cdot \rho(n)$ for all positive integers m and n. For part (ii) see e.g. [67, p. 209, Lemma 3].) Theorem 4.2 does not extend to either $\alpha(n)$ or $\beta(n)$. As a consequence of the classic convergence theorem for transition probabilities, any strictly stationary countable-state irreducible aperiodic Markov chain is absolutely regular. Such Markov chains exist for which the rate of convergence of $\alpha(n)$ (and hence also the rate for $\beta(n)$) to 0 is slower than exponential (see e.g. [30, Examples 1 and 2] or [51, p. 414, Corollary 1]). (By Theorem 4.2 such Markov chains cannot be ρ-mixing.) Of course every stationary *finite*-state irreducible aperiodic Markov chain is ψ-mixing (with exponential mixing rate).

A strictly stationary real Markov chain (X_k) is said to be a "Harris chain" if it has the Harris recurrence property: There exists a regular version of the conditional distribution of $(X_1, X_2, X_3, \ldots)$ given X_0 such that for every $x \in \mathbb{R}$, for every Borel subset $B \subset \mathbb{R}$ such that $P(X_0 \in B) > 0$, one has that $P(X_n \in B$ for infinitely many positive integers $n \mid X_0 = x) = 1$. (Thus every stationary countable-state irreducible Markov chain is a stationary Harris chain . Also, non-stationary Harris chains will not be discussed here.) It is well known that every stationary Harris chain has a well defined "period" $p \in \{1,2,3,\ldots\}$ (the chain is said to be "aperiodic" if $p = 1$). This fact and the next theorem can be seen (with a little work) from Orey [57, p. 13, Theorem 3.1; p. 23, Theorem 5.1; and p. 25, lines 9-13].

Theorem 4.3. (i) Every strictly stationary real aperiodic Harris chain is absolutely regular. (ii) More generally, for any strictly stationary real Harris chain, $\lim_{n\to\infty} \beta(n) = 1 - 1/p$ where p is the period.

A sequence (X_k) is said to be an "instantaneous function" of a real Markov chain (Y_k) if there is a Borel-measurable function $f\colon \mathbb{R} \to \mathbb{R}$ such that for each $k \in \mathbb{Z}$, $X_k = f(Y_k)$. As a consequence of Theorem 4.3(i), any instantaneous function of a stationary real aperiodic Harris chain is a stationary absolutely regular sequence. In [4] there are some stationary ϕ-mixing sequences which *cannot* be represented (on any probability space) as an instantaneous function of a stationary real Harris chain (periodic or aperiodic). It is apparently unknown whether any such sequences exist which are ψ-mixing or even 1-dependent.

Example 4.4. Rosenblatt [67, p. 214, line -3 to p. 215, line 13] presents a class of stationary real Markov chains which are ρ-mixing but not absolutely regular. (Consequently, they are not Harris chains.) His construction is based on "random rotations". Here we shall construct one of those examples in a different way, as an application of Theorem 3.1.

As a preliminary step, consider a stationary Markov chain $(W_k, k \in \mathbb{Z})$ with two states $\{0,1\}$, with invariant probability vector $(\frac{1}{2},\frac{1}{2})$, and with one-step transition probability matrix (p_{ij}) given by $p_{00} = p_{11} = 3/4$, $p_{10} = p_{01} = \frac{1}{4}$. By an induction argument, for each $n \geq 1$ the n-step transition probability matrix $(p_{ij}^{(n)})$ is given by $p_{00}^{(n)} = p_{11}^{(n)} = (1 + 2^{-n})/2$, $p_{10}^{(n)} = p_{01}^{(n)} = (1 - 2^{-n})/2$. A simple argument will show that for each $n \geq 1$,

(W_k) satisfies $\rho(n) = \rho(\sigma(W_0), \sigma(W_n)) = |\mathrm{Corr}(W_0, W_n)| = 2^{-n}$. Here the first inequality comes from Theorem 4.1 and the second from the elementary fact that every function of a two-state r.v. is automatically an *affine* function.

Now let $(W_k^{(j)}, k \in \mathbf{Z})$, $j = 1,2,3,\ldots$ be independent Markov chains, each having the same distribution as the Markov chain $(W_k, k \in \mathbf{Z})$ above. Define the sequence $(X_k, k \in \mathbf{Z})$ by $X_k := \Sigma_{j=1}^{\infty} 2^{-j} W_k^{(j)}$. Up to null sets, for each k, $\sigma(X_k) = \sigma(W_k^{(j)}, j = 1,2,\ldots)$ (i.e. from X_k one can calculate the value of $W_k^{(j)}$, $j = 1,2,3,\ldots$ a.s.). It follows that the sequence (X_k) is a Markov chain, and it is easy to see that it is strictly stationary. Further, (X_k) satisfies $\rho(n) = 2^{-n}$ for all $n \ge 1$, by Theorem 3.1 and the properties of (W_k) above; and hence (X_k) is ρ-mixing. In the framework of eqn. (2.1) one has that for each $n \ge 1$, by a simple calculation and the strong law of large numbers, $P(\lim_{J\to\infty} \Sigma_{j=1}^{J} W_0^{(j)} W_n^{(j)} = (\tfrac{1}{4})(1 + 2^{-n})) = 1$ and $Q(\lim_{J\to\infty} \Sigma_{j=1}^{J} W_0^{(j)} W_n^{(j)} = \tfrac{1}{4}) = 1$, and hence P and Q are mutually singular on $\sigma(X_0, X_n)$. Hence (X_k) satisfies $\beta(n) = 1$ for all $n \ge 1$, i.e. (X_k) fails to be absolutely regular. This completes Example 4.4.

5. MIXING CONDITIONS FOR GAUSSIAN SEQUENCES

For stationary real Gaussian sequences, a thorough discussion of the various mixing conditions is given by Ibragimov and Rozanov [48] [49, Chapters 4-5]. Theorem 5.1 here essentially just lists a few basic results from that discussion.

<u>Theorem 5.1</u>: Suppose $(X_k, k \in \mathbf{Z})$ is a (non-degenerate) stationary real Gaussian sequence. Then the following four statements hold:

(1) The following two conditions are equivalent:

(a) (X_k) is regular.

(b) (X_k) has an absolutely continuous spectral distribution function, and its spectral density f (defined on $[-\pi,\pi]$) satisfies $\int_{-\pi}^{\pi} \log f(\lambda) d\lambda > -\infty$.

(2) The following three conditions are equivalent:

(a) (X_k) is strongly mixing.

(b) (X_k) is ρ-mixing.

(c) The spectral density f of (X_k) can be expressed in the form $f(\lambda) = |P(e^{i\lambda})|^2 \exp[u(e^{i\lambda}) + \tilde{v}(e^{i\lambda})]$ where P is a polynomial, u and v are continuous real functions on the unit circle (in the complex plane), and $\tilde{v}$ is the conjugate function of v.

(3) The following two conditions are equivalent:

(a) (X_k) is absolutely regular.

(b) The spectral density f of (X_k) can be expressed in the form $f(\lambda) = |P(e^{i\lambda})|^2 \exp[\Sigma_{j=-\infty}^{\infty} a_j e^{ij\lambda}]$ (the sum converging in $L_2[-\pi,\pi]$) where P is a polynomial whose roots (if there are any) lie on the unit circle and $\Sigma_{j=-\infty}^{\infty} |j| \cdot |a_j|^2 < \infty$.

(4) The following four conditions are equivalent:

(a) (X_k) is ϕ-mixing.

(b) (X_k) is ψ-mixing.

(c) (X_k) is m-dependent.

(d) The spectral density f of (X_k) can be expressed in the form $f(\lambda) = |P(e^{i\lambda})|^2$ where P is a polynomial.

A few comments are in order. If f is the spectral density of a stationary real (not complex) Gaussian sequence then of course f is symmetric about 0. In connection with statement (2), one has (for Gaussian sequences) that $\rho(n) \leq 2\pi\cdot\alpha(n)$ and that $\rho(n)$ is identical to the supremum of $|\mathrm{Corr}(Y,Z)|$ taken over all finite linear combinations $Y = a_0X_0 + a_{-1}X_{-1} +\ldots+ a_{-M}X_{-M}$ and $Z = a_nX_n + a_{n+1}X_{n+1} +\ldots+ a_{n+N}X_{n+N}$ (see [52, Theorems 1 and 2]). In (2) the equivalence of (c) with (b) comes from the formulation of the Helson-Sarason theorem given in [70, p. 62]. From (2) and (3) we see that in order to construct a stationary real Gaussian sequence which is ρ-mixing but not absolutely regular, one can simply choose a spectral density which is positive and continuous but very "jagged", such as $f(\lambda) = \exp[\Sigma_{j=1}^{\infty} 2^{-j}\cos(2^{2j}\lambda)]$. For a stationary real ρ-mixing (or even absolutely regular) Gaussian sequence the spectral density need not be continuous or even bounded; consider an example with spectral density $f(\lambda) = \exp[\Sigma_{j=2}^{\infty}(j \log j)^{-1}\cos(j\lambda)]$ (which satisfies $\lim_{\lambda\to 0} f(\lambda) = +\infty$ by [77, p. 188, Theorem 2.15]). For more examples see [49, pp. 179-180]. For part (4) one can use the Wold decomposition theorem to show that any stationary real ϕ-mixing Gaussian sequence must be a moving average of i.i.d. Gaussian r.v.'s; one uses the fact that if Y and Z are jointly Gaussian r.v.'s with $\mathrm{Corr}(Y,Z) \neq 0$ then $\phi(\sigma(Y), \sigma(Z)) = 1$. To see this latter fact,

say in the case where $\mathrm{Corr}(Y,Z) > 0$, one can first note that $P(Z > q)$ becomes arbitrarily small as $q > 0$ becomes sufficiently large, and then for q fixed, $P(Z > q | Y > r)$ becomes arbitrarily close to 1 as $r > 0$ becomes sufficiently large.

Ibragimov and Rozanov [48] [49, p. 182, Lemma 17, and p. 190, Note 2] proved that every stationary Gaussian sequence satisfying $\Sigma_{n=1}^{\infty}\rho(2^n) < \infty$ has a continuous spectral density $f(\lambda)$; they derived for each $n \geq 1$ an upper bound on the "uniform error" of the "best" approximation of f by a trigonometric polynomial of degree $\leq n$. Their result introduced the (logarithmic) mixing rate $\Sigma_{n=1}^{\infty}\rho(2^n) < \infty$ into the literature, along with some of the techniques for handling this mixing rate. In central limit theory this mixing rate has turned out to be quite prominent for ρ-mixing (see e.g. [60] [58] [38] and the references therein).

6. SOME OTHER SPECIAL EXAMPLES

Here we shall briefly describe the strong mixing properties of a few stationary sequences that arise in other areas such as time series analysis, number theory, and interacting particle systems. (There will be a slight overlap with the Markov chains and Gaussian sequences studied in Sections 4 and 5.) In most of these examples we shall encounter strong mixing conditions with exponential mixing rates -- a context in which the known limit theory under strong mixing conditions usually applies very nicely. However, in Example 6.2 below, we shall also look at a well known simple stationary AR(1) process (autoregressive process of order 1) which *fails* to be strongly mixing.

Example 6.1. Suppose $(Z_k, k \in \mathbb{Z})$ is an i.i.d. sequence and the marginal distribution of Z_0 is absolutely continuous with a density which (to start with a few nice specific cases) is Gaussian, Cauchy, exponential, or uniform (on some interval). Suppose $a_1, a_2, a_3, \ldots$ is a sequence of real numbers with $|a_n| \to 0$ exponentially fast. Then the random sequence $(X_k, k \in \mathbf{Z})$ defined by

$$X_k := \sum_{j=0}^{\infty} a_j Z_{k-j} ,$$

is well defined, strictly stationary, and satisfies absolute regularity with exponential mixing rate. In particular this includes the cases where (X_k) is a stationary ARMA (autoregressive-mixed-moving-average) process based on the i.i.d. sequence (Z_k) given above. The conditions on (a_n) can be relaxed somewhat; if $a_n \to 0$ at a sufficiently fast polynomial rate, then (X_k) will still satisfy absolute regularity, but the mixing rate may be slower than

exponential. Of course all these statements apply to a much broader class of density functions for Z_0 than just the ones given above. For details see [61] and the references therein. As the next example will show, the above results in general do not carry over to the case where the distribution of Z_0 is discrete.

Example 6.2. The following example is well known; see e.g. [69, p. 267]. Suppose $(Z_k, k \in \mathbf{Z})$ is i.i.d. with $P(Z_k = 0) = P(Z_k = 1) = \frac{1}{2}$. Define the sequence $(X_k, k \in \mathbf{Z})$ by

$$X_k := (\tfrac{1}{2})Z_k + (\tfrac{1}{4})Z_{k-1} + (\tfrac{1}{8})Z_{k-2} + (\tfrac{1}{16})Z_{k-3} + \cdots$$

Then (X_k) is a strictly stationary AR(1) process; it can be represented by $X_k = (\frac{1}{2})X_{k-1} + (\frac{1}{2})Z_k$. For each k the r.v. X_k is uniformly distributed on the interval $[0,1]$. (Note that the digits in the binary expansion of X_k are $Z_k, Z_{k-1}, Z_{k-2}, \ldots$) For each k one also has that X_k is a (Borel-measurable) function of X_{k+1} (up to null sets): X_k = (fractional part of $2X_{k+1}$) a.s. As a consequence of all this, for each $n \geq 1$, X_0 is a (Borel-measurable) function of X_n (by induction) and hence

$$\begin{aligned}\alpha(n) &\geq \alpha(\sigma(X_0), \sigma(X_n)) \geq \alpha(\sigma(X_0), \sigma(X_0)) \\ &\geq P(X_0 < \tfrac{1}{2}) - [P(X_0 < \tfrac{1}{2})]^2 = \tfrac{1}{4} .\end{aligned}$$

Hence (X_k) *fails* to be strongly mixing; in fact $\alpha(n) = \frac{1}{4}$ for all $n \geq 1$ by (1.11). Indeed, even though (X_k) is regular, its future tail σ-field $\bigcap_{n=1}^{\infty} F_n^{\infty}$ is non-trivial (it coincides with $F_{-\infty}^{\infty}$ up to null sets). Of course one can make this example symmetric about 0 by replacing Z_k by $Z_k - \frac{1}{2}$.

Example 6.3. This is another well known example, related to number theory. For every irrational number $x \in (0,1)$ there exists a unique sequence of positive integers $x_1, x_2, x_3, \ldots$ such that the following "continued fraction" expansion holds:

$$x = \cfrac{1}{x_1 + \cfrac{1}{x_2 + \cfrac{1}{x_3 + \ddots}}}$$

Suppose we impose the Gauss probability measure on [0,1], namely the measure which is absolutely continuous with density $f(x) := (\log 2)^{-1} \cdot (1+x)^{-1}$. In this context the sequence $(x_1, x_2, x_3, \ldots)$ is a strictly stationary sequence of random variables, and it is ψ-mixing with exponential mixing rate (see [62, p. 450, Corollary 1]). For applications of limit theorems under strong mixing conditions to number theory (including this continued fraction expansion), see [62] [63].

Example 6.4. There has been a lot of research on the ergodic-theoretic properties of mappings $T: [0,1] \to [0,1]$. See e.g. [55] as well as the references given below. Example 6.3 above fits into this framework in a natural way (see e.g. [62, pp. 448-449]). Here we shall just describe one other simple example. Suppose $2^{1/2} < \lambda \le 2$. Consider the mapping $T: [0,1] \to [0,1]$ defined by

$$T(x) := \begin{cases} \lambda x + 2 - \lambda & \text{if } 0 \le x \le 1 - 1/\lambda \\ -\lambda x + \lambda & \text{if } 1 - 1/\lambda \le x \le 1 \end{cases}$$

The graph of this function looks somewhat like an inverted, chopped letter V with apex at the point $(1 - 1/\lambda, 1)$. By [53, Theorem 1] there exists on

[0,1] an absolutely continuous probability measure μ which is T-invariant. In [45, Theorem 1(vi)] a "canonical" method is described for defining μ. Under this measure μ the transformation T is "weak mixing" by [10, Theorem 2(ii)]. (We need not give the definition of "weak mixing" here.) Now let $I_1, I_2, \ldots, I_M$ be an arbitrary partition of [0,1] into finitely many intervals, and define on $([0,1], \mathcal{B}_{[0,1]}, \mu)$ the (strictly stationary) sequence $(X_k, k=1,2,3,\ldots)$ by

$$X_k(x) := m \quad \text{if} \quad T^k(x) \in I_m, \; m=1,2,\ldots, M.$$

By [45, p. 132, lines 4-6], (X_k) is absolutely regular with exponential mixing rate.

<u>Example 6.5</u>. Gibbs measures have sometimes been used in the study of interacting particle systems. We shall discuss Gibbs measures in just the simplest context. Suppose $\Phi: \{0,1\}^{\mathbb{Z}} \to \mathbb{R}$ is a function with the following restriction on its "variation": $\exists\, a>0$, $\exists\, C>0$, such that for all $m=1,2,3,\ldots$

$$[\sup\{|\Phi(x) - \Phi(y)|: x,y \in \{0,1\}^{\mathbb{Z}} \text{ such that } x_k = y_k \;\; \forall k = -m, -m+1, \ldots, m\}] \le Ce^{-am}$$

(where the k^{th} coordinate of any $x \in \{0,1\}^{\mathbb{Z}}$ is denoted x_k). Then there exists a unique shift-invariant probability measure μ on $\{0,1\}^{\mathbb{Z}}$ with the following property: $\exists\, C_1>0, \exists\, C_2>0, \exists\, q \in \mathbb{R}$ such that for all $x \in \{0,1\}^{\mathbb{Z}}$, for all $m \ge 1$,

$$C_1 \le \frac{\mu\{y: \; y_k = x_k \text{ for all } k=0,1,\ldots, m-1\}}{\exp[-qm + \Sigma_{0\le k\le m-1}\Phi(T^k(x))]} \le C_2$$

where T is the usual shift operator on $\{0,1\}^{\mathbb{Z}}$. This measure μ is called the

"Gibbs measure" based on the function Φ. (For details see e.g. [9] [35].) On the probability space $(\{0,1\}^{\mathbf{Z}}, \mathcal{B}^{\mathbf{Z}}_{\{0,1\}}, \mu)$ define the (strictly stationary) random sequence $(X_k, k \in \mathbf{Z})$ by $X_k(\omega) := \omega_k$. One can interpret $X_k = 0$ resp. 1 to mean that the k^{th} site is empty resp. occupied by a particle. This sequence (X_k) is ψ-mixing with exponential mixing rate (see [9, p. 24] or [35]).

7. THE BEHAVIOR OF THE DEPENDENCE COEFFICIENTS

First we shall examine the possible limit values of the dependence coefficients.

Theorem 7.1: If $(X_k, k \in \mathbf{Z})$ is strictly stationary and mixing in the ergodic theoretic sense, then the following three statements hold:

(i) Either $\beta(n) \to 0$ as $n \to \infty$ or $\beta(n) = 1$ for all $n \geq 1$.

(ii) Either $\phi(n) \to 0$ as $n \to \infty$ or $\phi(n) = 1$ for all $n \geq 1$.

(iii) Either $\psi(n) \to 0$, $\psi(n) \to 1$, or $\psi(n) = \infty$ for all $n \geq 1$.

Statements (i) and (ii) can be found in [13, Theorem 1] and [11, Theorem 1], and statement (iii) is a trivial consequence of [14, Theorem 1]. Statement (i) is a slight extension of an earlier result of Volkonskii and Rozanov [74, Theorem 4.1]. Statement (ii) was previously known for stationary Markov chains (see [67, p. 209, Lemma 3]) and of course for stationary Gaussian sequences (see Section 6). Theorem 7.1 does not extend to either $\alpha(n)$ or $\rho(n)$. Instead, for stationary regular sequences, lim $a(n)$ can be any value in $[0,\frac{1}{4}]$ and lim $\rho(n)$ can be any value in $[0,1]$. (See eqn. (1.11) and [12, Theorem 6]; regularity was not mentioned in this theorem, but is an elementary property of the construction given in the proof.) Berbee [2, Theorem 2.1] proved an analog of Theorem 7.1 for stationary sequences which are ergodic but not assumed to be mixing:

Theorem 7.2 (Berbee): If $(X_k, k \in \mathbf{Z})$ is strictly stationary and ergodic, then $\lim_{n\to\infty} \beta(n) = 1 - 1/p$ for some $p \in \{1,2,3,\ldots\} \cup \{\infty\}$. If this p satisfies

$2 \le p < \infty$, then letting T denote the usual shift operator (on events in $F_{-\infty}^{\infty}$), the invariant σ-field of T^p is identical to each tail σ-field of (X_k) (up to null sets) and is purely atomic with exactly p atoms, each having probability $1/p$ (the p atoms are A, TA, $T^2A, \ldots, T^{p-1}A$ where A is any one of the atoms), and conditional on any one of these atoms the sequence of random vectors $(Y_k, k \in \mathbb{Z})$ defined by $Y_k := (X_{(k-1)p+1}, X_{(k-1)p+2}, \ldots, X_{kp})$ is strictly stationary and satisfies the absolute regularity condition.

Theorem 4.3(ii) is in essence a special case of Theorem 7.2. Also, as a simple corollary of Theorem 7.2 (after one applies Theorem 7.1(ii)(iii) to the sequence (Y_k) there if $2 \le p < \infty$) one has the following additional properties of strictly stationary ergodic sequences (X_k):

(i) $\phi(n) \to 1 - 1/p$ for some $p \in \{1,2,3,\ldots\} \cup \{\infty\}$,

(ii) $\psi(n) \to p - 1$ for some $p \in \{1,2,3,\ldots\} \cup \{\infty\}$.

(In particular, for example, if $\lim \beta(n) = 1 - 1/p$ for some (finite) positive integer p, then either $\lim \phi(n) = 1 - 1/p$ for the same p or else $\phi(n) = 1$ for all n.)

For strictly stationary sequences there is essentially no restriction on the mixing rates for the mixing conditions being discussed here. See e.g. [51, Theorems 2,3, and 4], [49, pp. 181-190], [11, Theorem 2], [12, Theorem 6], and [14, Theorem 2]. Also, for most mixing rates used in the literature (in particular exponential, polynomial, or logarithmic), the mixing conditions being discussed here can all hold with essentially the same given rate; this is a consequence of the following theorem which (because of eqns. (1.7) and (1.8)) is just [18, Theorem 1]:

<u>Theorem 7.3</u>: Suppose $g: [0,\infty) \to (0,\infty)$ is a positive continuous non-increasing function such that $\lim_{x\to\infty} g(x) = 0$, $g(0) \le 1/24$, and $\log g$ is convex on $[0,\infty)$. Then there exists a strictly stationary sequence (X_k) such that for every $n \ge 1$, $(\frac{1}{4})g(n) \le \alpha(n), \beta(n), \rho(n), \phi(n), \psi(n) \le 8g(n)$.

(The conditions on g here are of course somewhat redundant.) Under fewer restrictions on g, Theorem 7.3 was already shown for non-stationary sequences by Kesten and O'Brien [51, Theorem 1].

Theorems 7.1 and 7.2 do not extend to non-stationary sequences. In limit theory for non-stationary sequences, conditions such as "$\psi(n) < \infty$ for some n" or "$\phi(n) < 1$ for some n" are sometimes useful (see e.g. [26] [58]). Note that if $\psi(n) < \infty$ then (for the same n) $\phi(n) < 1$. (In fact it is not hard to show that $\phi(\mathcal{A},\mathcal{B}) < 1$ if either $\sup P(A\cap B)/[P(A)P(B)] < \infty$ or $\inf P(A\cap B)/[P(A)P(B)] > 0$, the sup and inf being taken over all $A \in \mathcal{A}$ and $B \in \mathcal{B}$ with $P(A) > 0$ and $P(B) > 0$.) For non-stationary sequences, conditions such as $\psi(n) < \infty$ or $\phi(n) < 1$ do not impose any essential restrictions on the moments of the r.v.'s X_k or on the rate at which $\alpha(n)$ or $\beta(n)$ or $\rho(n)$ might perhaps converge to 0, nor do they preclude *second-order* stationarity. Defining $\phi_{rev}(n) := \sup_{J\in\mathbf{Z}} \phi_{rev}(\mathcal{F}_{-\infty}^{J}, \mathcal{F}_{J+n}^{\infty})$ for each $n = 1,2,\ldots,$ one has the following theorem:

<u>Theorem 7.4</u>: Suppose $d_1, d_2, d_3, \ldots$ is a non-increasing sequence of positive numbers such that $\lim_{n\to\infty} d_n = 0$. Suppose $0 < c \le 1$. Then there exists a (not strictly stationary) sequence $(X_k, k \in \mathbf{Z})$ with the following eight properties:

(i) For all $k \in \mathbb{Z}$, $EX_k = 0$ and $EX_k^2 = 1$.

(ii) For all $k, \ell \in \mathbb{Z}$ with $k \neq \ell$, $EX_k X_\ell = 0$.

(iii) For all $k \in \mathbb{Z}$, $|X_k| \leq 2$ a.s.

(iv) For all $n \geq 1$, $\psi(n) = c$.

(v) For all $n \geq 1$, $\phi(n) = \phi_{rev}(n) = c/2$.

(vi) $\rho(n) \overset{\cup}{\cap} d_n^{\frac{1}{2}}$ as $n \to \infty$

(vii) $\beta(n) \overset{\cup}{\cap} d_n$ as $n \to \infty$

(viii) $\alpha(n) \overset{\cup}{\cap} d_n$ as $n \to \infty$

Here the notation $a \overset{\cup}{\cap} b$ means that $a = O(b)$ and $b = O(a)$. This theorem apparently has not been published anywhere; its proof will be given here, based on an argument used by Kesten and O'Brien [51, Theorem 1].

First some preliminary calculations are needed. Suppose $0 < c \leq 1$ and $0 < d \leq \frac{1}{4}$. Suppose U and V are r.v.'s, each with state space $\{-2, -1, 1, 2\}$ and marginal probability vector $[d, \frac{1}{2} - d, \frac{1}{2} - d, d]$, and with joint probability function (f_{ij}), where $f_{ij} := P(U = i, V = j)$, given by the matrix

$$\begin{bmatrix} f_{-2,-2}, & f_{-2,-1}, & f_{-2,1}, & f_{-2,2} \\ f_{-1,-2}, & f_{-1,-1}, & f_{-1,1}, & f_{-1,2} \\ f_{1,-2}\,, & f_{1,-1}\,, & f_{1,1}\,, & f_{1,2} \\ f_{2,-2}\,, & f_{2,-1}\,, & f_{2,1}\,, & f_{2,2} \end{bmatrix} :=$$

$$:= \begin{bmatrix} d^2 & d(\frac{1}{2} - d)(1 + c) & d(\frac{1}{2} - d)(1 - c) & d^2 \\ d(\frac{1}{2} - d)(1 - c) & (\frac{1}{2} - d)^2 & (\frac{1}{2} - d)^2 & d(\frac{1}{2} - d)(1 + c) \\ d(\frac{1}{2} - d)(1 + c) & (\frac{1}{2} - d)^2 & (\frac{1}{2} - d)^2 & d(\frac{1}{2} - d)(1 - c) \\ d^2 & d(\frac{1}{2} - d)(1 - c) & d(\frac{1}{2} - d)(1 + c) & d^2 \end{bmatrix}$$

By elementary calculations one can show that $\psi(\sigma(U), \sigma(V)) = c$, $\phi(\sigma(U), \sigma(V)) = \phi_{rev}(\sigma(U), \sigma(V)) = c(\frac{1}{2} - d)$, $\beta(\sigma(U), \sigma(V)) = 4cd(\frac{1}{2} - d)$, and $\alpha(\sigma(U), \sigma(V)) \geq P(U=1 \text{ or } 2, V=-2 \text{ or } 1) - P(U=1 \text{ or } 2)P(V=-2 \text{ or } 1) = 2cd(\frac{1}{2} - d)$. By the first equality in (1.7), in fact $\alpha(\sigma(U), \sigma(V)) = 2cd(\frac{1}{2} - d)$. Also, by eqn. (1.12), $\alpha_{\frac{1}{2},\frac{1}{2}}(\mathcal{A},\mathcal{B}) \leq [\alpha(\mathcal{A},\mathcal{B})\cdot\psi(\mathcal{A},\mathcal{B})]^{\frac{1}{2}} \leq cd^{\frac{1}{2}}$. Since U and V each have only four states, $\rho(\sigma(U), \sigma(V)) \leq 16\alpha_{\frac{1}{2},\frac{1}{2}}(\mathcal{A},\mathcal{B}) \leq 16cd^{\frac{1}{2}}$. Also, $\rho(\sigma(U), \sigma(V)) \geq \mathrm{Corr}(I_{\{U=2\}}, I_{\{V=1\}}) \geq (c/2)d^{\frac{1}{2}}$ (using the fact that $d \leq \frac{1}{4}$). Also, $EU = EV = EUV = 0$, $EU^2 = EV^2 > 1$, and $\|U\|_\infty = \|V\|_\infty = 2$.

Now we mimic the construction in [51, Theorem 1]. For each $m = 1,2,3,\ldots$ let $(X_k,\ k = m^2,\ m^2+m)$ be a random vector with the same distribution as the random vectors (U,V) above, in the case where the parameters are c and d_m from the hypothesis of Theorem 7.4. We assume without loss of generality that $d_m \leq \frac{1}{4}$ for all m. For all integers k that are *not* of the form m^2 or m^2+m, $m = 1,2,\ldots$, let X_k be a r.v. such that $P(X_k = 1) = P(X_k = -1) = \frac{1}{2}$. We impose the additional restriction that (X_1,X_2), (X_4,X_6), (X_9,X_{12}), $(X_{16},X_{20}),\ldots$ and $\ldots X_{-2}, X_{-1}, X_0, X_3, X_5, X_7, X_8, X_{10}, X_{11}, X_{13},\ldots$ are independent. Following the argument in [51, Theorem 1] we have that for each $n = 1,2,3,\ldots$

$$\alpha(n) = \sup_{m\geq n}\alpha(\sigma(X_{m^2}), \sigma(X_{m^2+m}))$$

and the analogous statement holds for the other dependence coefficients. Using the calculations in the preceding paragraph, one can show that the sequence (X_k) constructed here satisfies all properties (i)-(viii) in Theorem 7.4, except that $EX_k^2 > 1$ for $k = m^2,\ m^2+m$, $m = 1,2,\ldots$. Dividing each X_k by its standard deviation, we obtain a new sequence (X_k) satisfying all properties in Theorem 7.4. This completes the proof.

8. APPROXIMATION OF MIXING SEQUENCES BY OTHER RANDOM SEQUENCES

In proving limit theorems for a given sequence of random variables satisfying strong mixing conditions, it is often useful to approximate the sequence by another random sequence with certain properties. Here we shall give some pertinent references on such techniques.

The technique of directly approximating a mixing sequence by a sequence of martingale differences was introduced by Gordin [40]. It has been exploited by many people (see e.g. [64] [42] as well as the exposition of it in [50, pp. 127-131]). A long-standing, previously unnoticed error in one particular application of Gordin's technique was pointed out by Herrndorf [43] (see [20] for further details).

The technique of directly approximating a mixing sequence by a sequence of independent random variables was introduced by Berkes and Philipp [6][7]. This technique is a versatile one which often allows one to handle very slow (logarithmic) mixing rates (see e.g. [7] [28] [16]), other types of conditions of weak dependence besides the strong mixing conditions being discussed here (see e.g. [36]), and also random variables taking their values in general Banach spaces (see e.g. [33] and the references therein). This technique works particularly nicely with the absolute regularity condition (see [1, Corollary 4.2.5] [3] [23] [33]). Dehling [31] exposed a weakness in this technique under just the strong mixing condition (or even, by the same argument, the ρ-mixing condition) when the random variables are taking their values in general Banach spaces.

For proving theorems in renewal theory, "coupling" techniques are often useful. Berbee [1, p. 104, Theorem 4.4.7] characterized the strictly stationary absolutely regular sequences in terms of a coupling property:

Theorem 8.1 (Berbee): A strictly stationary sequence $(X_k, k \in \mathbb{Z})$ of random variables is absolutely regular if and only if there exists a probability space with sequences $(X_k', k \in \mathbb{Z})$ and $(X_k'', k \in \mathbb{Z})$, each having the same distribution as $(X_k, k \in \mathbb{Z})$, such that (i) $(X_k', k \leq 0)$ and $(X_k'', k \in \mathbb{Z})$ are independent and (ii) $P(\exists n \geq 1$ such that $X_k' = X_k''$ for all $k \geq n) = 1$.

Berbee [1, p. 106] also explains that if (X_k) is absolutely regular (i.e. satisfies $\beta(n) \to 0$), then in the context of Theorem 8.1 the inequality

$$P(\exists k \geq n \text{ such that } X_k' \neq X_k'') \geq \beta(n) \tag{8.1}$$

automatically holds for all $n \geq 1$, and that there exists an "optimal" coupling in which equality in eqn. (8.1) is achieved simultaneously for all $n \geq 1$. This is analogous to Griffeath's [41] well known "maximal coupling" result for Markov chains.

Acknowledgements: This paper arose from conversations with a lot of people. Special thanks are due to M. Denker (who suggested Examples 6.3, 6.4, and 6.5), M. Peligrad, and M. Rosenblatt for their comments.

REFERENCES

1. H. Berbee. Random walks with stationary increments and renewal theory. Mathematical Centre, Amsterdam, 1979.

2. H. Berbee. Weak Bernoulli processes, periodicity and renewal theory on the integers. Report no. 167. Mathematical Centre, Amsterdam, 1981.

3. H. Berbee. Convergence rates in the strong law for bounded mixing sequences. Report MS-R8412. Mathematical Centre, Amsterdam, 1984.

4. H. Berbee and R.C. Bradley. A limitation of Markov representation for stationary processes. Stochastic Proc. Appl. 18 (1984) 33-45.

5. J. Bergh and J. Löfström. Interpolation spaces. Springer-Verlag, Berlin, 1976.

6. I. Berkes and W. Philipp. An almost sure invariance principle for the empirical distribution function of mixing random variables. Z. Wahrsch. Verw. Geb. 41 (1977) 115-137.

7. I. Berkes and W. Philipp. Approximation theorems for independent and weakly dependent random vectors. Ann. Probab. 7 (1979) 29-54.

8. J.R. Blum, D.L. Hanson, and L.H. Koopmans. On the strong law of large numbers for a class of stochastic processes. Z. Wahrsch. Verw. Geb. 2 (1963) 1-11.

9. R. Bowen. Equilibrium states and the ergodic theory of Anosov diffeomorphisms. Lecture Notes in Mathematics 470. Springer-Verlag, Berlin, 1975.

10. R. Bowen. Bernoulli maps of the interval. Israel J. Math. 28 (1977) 161-168.

11. R.C. Bradley. On the ϕ-mixing condition for stationary random sequences. Duke Math. J. 47 (1980) 421-433.

12. R.C. Bradley. Central limit theorems under weak dependence. J. Multivar. Anal. 11 (1981) 1-16.

13. R.C. Bradley. Absolute regularity and functions of Markov chains. Stochastic Proc. Appl. 14 (1983) 67-77.

14. R.C. Bradley. On the ψ-mixing condition for stationary random sequences. Trans. Amer. Math. Soc. 276 (1983) 55-66.

15. R.C. Bradley. Equivalent measures of dependence. J. Multivar. Anal. 13 (1983) 167-176.

16. R.C. Bradley. Approximation theorems for strongly mixing random variables. Michigan Math. J. 30 (1983) 69-81.

17. R.C. Bradley. On a very weak Bernoulli condition. Stochastics 13 (1984) 61-81.

18. R.C. Bradley. Identical mixing rates. Preprint, 1984.

19. R.C. Bradley. A bilaterally deterministic ρ-mixing stationary random sequence. Tech. Rept. No. 91, Center for Stochastic Processes, Univ. of North Carolina, Chapel Hill, 1985.

20. R.C. Bradley. On the central limit question under absolute regularity. Ann. Probab. (1985), to appear.

21. R.C. Bradley and W. Bryc. Multilinear forms and measures of dependence between random variables. J. Multivar. Anal. (1985), to appear.

22. R.C. Bradley, W. Bryc, and S. Janson. Some remarks on the foundations of measures of dependence. Tech. Rept., Center for Stochastic Processes, University of North Carolina, Chapel Hill, 1985. (In preparation).

23. W. Bryc. On the approximation theorem of I. Berkes and W. Philipp. Demonstratio Math. 15 (1982) 807-816.

24. A.V. Bulinskii. On measures of dependence close to the maximal correlation coefficient. Soviet Math. Dokl. 30 (1984) 249-252.

25. R. Cogburn. Asymptotic properties of stationary sequences. Univ. Calif. Publ. Statist. 3 (1960) 99-146.

26. H. Cohn. On a class of dependent random variables. Revue Roumanie Pures et Appl. 10 (1965) 1593-1606.

27. P. Csaki and J. Fischer. On the general notion of maximal correlation. Magyar Tud. Akad. Mat. Kutato Int. Kozl. 8 (1963) 27-51.

28. A.R. Dabrowski. A note on a theorem of Berkes and Philipp for dependent sequences. Statist. and Probab. Letters 1 (1982) 53-55.

29. R.A. Davis. Stable limits for partial sums of dependent random variables. Ann. Probab. 11 (1983) 262-269.

30. Y.A. Davydov. Mixing conditions for Markov chains. Theory Probab. Appl. 18 (1973) 312-328.

31. H. Dehling. A note on a theorem of Berkes and Philipp. Z. Wahrsch. Verw. Geb. 62 (1983) 39-42.

32. H. Dehling, M. Denker, and W. Philipp. Versik processes and very weak Bernoulli processes with summable rates are independent. Proc. Amer. Math. Soc. 91 (1984) 618-624.

33. H. Dehling and W. Philipp. Almost sure invariance principles for weakly dependent vector-valued random variables. Ann. Probab. 10 (1982) 689-701.

34. M. Denker and G. Keller. On U-statistics and V. Mises' statistics for weakly dependent processes. Z. Wahrsch. Verw. Geb. 64 (1983) 505-522.

35. M. Denker and W. Philipp. Approximation by Brownian motion for Gibbs measures and flows under a function. Ergod. Th. and Dynam. Sys. 4 (1984).

36. E. Eberlein. Strong approximation of very weak Bernoulli processes, Z. Wahrsch. Verw. Geb. 62 (1983) 17-37.

37. E. Eberlein. Weak convergence of partial sums of absolutely regular sequences. Statist. and Probab. Letters 2 (1984) 291-293.

38. M. Falk. On the convergence of spectral densities of arrays of weakly stationary processes. Ann. Probab. 12 (1984) 918-921.

39. H. Gebelein. Das Statistische Problem der Korrelation als Variations-- und Eigenvertproblem und sein Zusammenhang mit der Ausgleichungsrechnung. Z. Angew. Math. Mech. 21 (1941) 364-379.

40. M.I. Gordin. The central limit theorem for stationary processes. Soviet Math. Dokl. 10 (1969) 1174-1176.

41. D. Griffeath. A maximal coupling for Markov chains. Z. Wahrsch. Verw. Geb. 31 (1975) 95-106.

42. P. Hall and C.C. Heyde. Martingale Limit Theory and Its Application. Academic Press, New York, 1980.

43. N. Herrndorf. Stationary strongly mixing sequences not satisfying the central limit theorem. Ann. Probab. 11 (1983) 809-813.

44. H.O. Hirschfeld. A connection between correlation and contingency. Proc. Camb. Phil. Soc. 31 (1935) 520-524.

45. F. Hofbauer and G. Keller. Ergodic properties in invariant measures for piecewise monotonic transformations. Math. Z. 180 (1982) 119-140.

46. I.A. Ibragimov. Some limit theorems for strictly stationary stochastic processes. Dokl. Akad. Nauk SSSR 125 (1959) 711-714. (In Russian).

47. I.A. Ibragimov and Y.V. Linnik. Independent and Stationary Sequences of Random Variables. Wolters-Noordhoff, Groningen, 1971.

48. I.A. Ibragimov and Y.A. Rozanov. Gaussian Random Processes. Nauka, Moscow, 1970. (In Russian).

49. I.A. Ibragimov and Y.A. Rozanov. Gaussian Random Processes. Springer-Verlag, Berlin, 1978.

50. M. Iosifescu. Recent advances in mixing sequences of random variables. In: Third international summer school on probability theory and mathematical statistics, Varna 1978, pp. 111-138. Publishing House of the Bulgarian Academy of Sciences, Sofia, 1980.

51. H. Kesten and G.L. O'Brien. Examples of mixing sequences. Duke Math. J. 43 (1976) 405-415.

52. A.N. Kolmogorov and Y.A. Rozanov. On strong mixing conditions for stationary Gaussian processes. Theory Probab. Appl. 5 (1960) 204-208.

53. A. Lasota and J.A. Yorke. On the existence of invariant measures for piecewise monotonic transformations. Trans. Amer. Math. Soc. 186 (1973) 481-488.

54. M.R. Leadbetter, G. Lindgren, and H. Rootzen. Extremes and Related Properties of Random Sequences and Processes. Springer-Verlag, Berlin, 1983.

55. F. Ledrappier. Some properties of absolute continuous invariant measures on an interval. Ergod. Th. and Dynam. Sys. 1 (1981) 77-93.

56. B./ Lifshits. Invariance principle for weakly dependent variables. Theory Probab. Appl. 29 (1984) 33-40.

57. S. Orey. Lecture notes on limit theorems for Markov chain transition probabilities. Mathematical Studies 34. Van Nostrand Reinhold, New York, 1971.

58. M. Peligrad. Invariance principles for mixing sequences of random variables. Ann. Probab. 10 (1982) 968-981.

59. M. Peligrad. A note on two measures of dependence and mixing sequences. Adv. Appl. Probab. 15 (1983) 461-464.

60. M. Peligrad. Recent advances in the central limit theorem and its invariance principle for mixing sequences of random variables. (In preparation.)

61. T.D. Pham and L.T. Tran. Some mixing properties of time series models. Stochastic Proc. Appl. 19 (1985) 297-303.

62. W. Philipp. Some metrical theorems in number theory II. Duke Math. J. 37 (1970) 447-458.

63. W. Philipp. Mixing sequences of random variables and probabilistic number theory. Amer. Math. Soc. Mem. No. 114, 1971.

64. W. Philipp and W. Stout. Almost sure invariance principles for partial sums of weakly dependent random variables. Amer. Math. Soc. Mem. No. 161, 1975.

65. M.S. Pinsker. Information and Information Stability of Random Variables and Processes. Holden-Day, San Francisco, 1964.

66. M. Rosenblatt. A central limit theorem and a strong mixing condition. Proc. Nat. Acad. Sci. USA 42 (1956) 43-47.

67. M. Rosenblatt. Markov Processes, Structure and Asymptotic Behavior. Springer-Verlag, Berlin, 1971.

68. M. Rosenblatt. Uniform ergodicity and strong mixing. Z. Wahrsch. Verw. Geb. 24 (1972) 79-84.

69. M. Rosenblatt. Linear processes and bispectra. J. Appl. Probab. 17 (1980) 265-270.

70. D. Sarason. An addendum to 'Past and Future'. Math. Scand. 30 (1972) 62-64.

71. P. Shields. The Theory of Bernoulli Shifts. Univ. of Chicago Press, Chicago, 1973.

72. E.M. Stein and G. Weiss. An extension of a theorem of Marcinkiewicz and some of its applications. J. Math. Mech. 8 (1959) 263-284.

73. V.A. Volkonskii and Y.A. Rozanov. Some limit theorems for random functions I. Theory Probab. Appl. 4 (1959) 178-197.

74. V.A. Volkonskii and Y.A. Rozanov. Some limit theorems for random functions II. Theory Probab. Appl. 6 (1961) 186-198.

75. C.S. Withers. Central limit theorems for dependent random variables I. Z. Wahrsch. Verw. Geb. 57 (1981) 509-534.

76. H.S. Witsenhausen. On sequences of pairs of dependent random variables. SIAM J. Appl. Math. 28 (1975) 100-113.

77. A. Zygmund. Trigonometric Series, Vol. I. Cambridge Univ. Press, Cambridge, 1959.

END

FILMED

10-85

DTIC